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A Note on the Computation of Diffuse Reflection Functions for Spherical Shells

It is shown that recent computational investigations of Bellman *et al.* on the diffuse reflection function for a spherical shell are invalid. The system of equations for the total reflection function considered by them admits two solutions, and the character of the numerical methods employed was such as to select the spurious solution. This solution was incorrectly identified with the diffuse reflection function.

1. Introduction

The method of invariant imbedding, which has been extensively applied to problems of radiative transfer in plane parallel media, has recently been extended to media with spherical symmetry [1–5]. One of the principal objects of this method is to determine the diffuse reflection function, which relates the incident and diffusely emergent intensities on the outer boundary of the medium. Bellman *et al.* [6, 7] have presented numerical results for the diffuse reflection function for a spherical shell consisting of an isotropically scattering sphere with a totally absorbing spherical core. The purpose of this paper is to show that the methods used by them are invalid, because the system of equations they have chosen does not have a unique solution and their computational procedures pick out the spurious solution.

In particular, the system of equations employed by them refers to the *total* reflection function rather than to the diffuse reflection function. This function consists of the well-behaved diffuse part plus a singular direct part, which contains a generalized function of Dirac type. It will be shown that this system also admits another solution, which is well-behaved, and it is this solution that is found computationally. This well-behaved solution differs in general from the desired diffuse part of the singular solution, but was incorrectly identified as such in [6, 7].

2. The Total and Diffuse Reflection Functions for a Sphere

The notation of Bellman, Kagiwada, and Kalaba [7] will be used with minor extensions. The total reflection function $S(z, v, u)$ is defined so that the incident and emergent intensities on the outer boundary of the sphere of radius z are related by

$$I(z, \nu) = \frac{1}{2\nu} \int_0^1 S(z, \nu, u)\, I(z, -u)\, du. \tag{2.1}$$

Here $I(z, \nu)$ is the specific intensity of radiation at radius z making an angle of cosine ν with the radius vector. For the case considered in [7] of an isotropically scattering spherical shell with a totally absorbing core of radius a, the system satisfied by S is

$$\begin{gathered}\frac{\partial S}{\partial z} + \frac{1-\nu^2}{z\nu}\frac{\partial S}{\partial \nu} + \frac{1-u^2}{zu}\frac{\partial S}{\partial u} - \frac{\nu^2+u^2}{z\nu^2u^2} S + \sigma\left(\frac{1}{\nu}+\frac{1}{u}\right) S \\ = \sigma\lambda\left[1 + \frac{1}{2}\int_0^1 S(z, \nu, u')\frac{du'}{u'}\right]\left[1 + \frac{1}{2}\int_0^1 S(z, \nu', u)\frac{d\nu'}{\nu'}\right], \qquad z > a, \\ S(a, \nu, u) = 0. \end{gathered} \tag{2.2}$$

This equation differs slightly from that of [7] in that the scattering coefficient σ appears explicitly, whereas there it was taken as unity. Also a typographical error in the last integral has been corrected.

In dealing with this problem it is convenient to separate emergent intensities into a *direct* part that has suffered no acts of scattering and the remaining *diffuse* part. The direct part emergent at radius z and at $\nu > 0$ is

$$I(z, -\nu)\, e^{-2\sigma z\nu} H(\nu_c - \nu), \tag{2.3}$$

where H denotes the unit Heaviside function, which is unity for positive values of the argument and vanishes otherwise. The quantity ν_c is a critical direction cosine defined by $\nu_c = [1 - (a/z)^2]^{1/2}$. This follows from simple geometrical considerations (see Fig. 1). Any incident ray with direction cosine $(-\nu)$ such that $\nu > \nu_c$ will intercept the absorbing core and will not contribute a direct part to the emergent intensities, while for $\nu < \nu_c$ the ray contributes to the direct part emergent at direction cosine $(+\nu)$, but is reduced by the attenuation factor $\exp[-2\sigma z\nu]$ due to scattering suffered along the path of length $2z\nu$.

A diffuse reflection function $\tilde{S}(z, \nu, u)$ may now be defined by removing this direct part:

$$I(z, \nu) = e^{-2\sigma z\nu} H(\nu_c - \nu)\, I(z, -\nu) + \frac{1}{2\nu}\int_0^1 \tilde{S}(z, \nu, u)\, I(z, -u)\, du. \tag{2.4}$$

It follows that

$$S(z, \nu, u) = \tilde{S}(z, \nu, u) + 2\nu e^{-2\sigma z\nu} H(\nu_c - \nu)\, \delta(\nu - u), \tag{2.5}$$

where δ denotes the Dirac δ-function. The diffuse reflection function is an ordinary function, so that the total reflection function is a generalized function with a δ-function singularity.

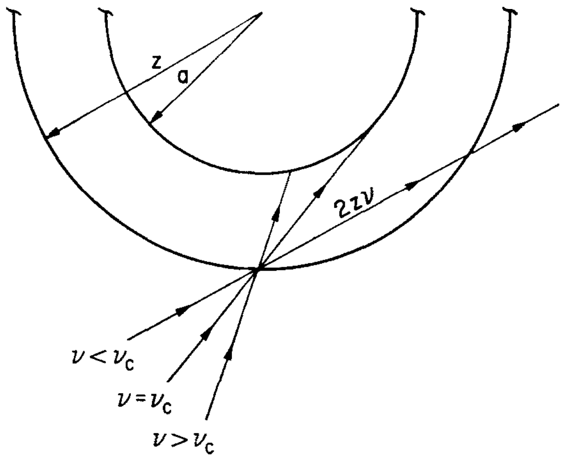

FIG. 1. Geometry of the direct rays. For $\nu > \nu_c$ the ray intercepts the absorbing core. For $\nu < \nu_c$ the ray eventually emerges, attenuated, from the scattering region and contributes to the total reflection function.

The system satisfied by the diffuse reflection function $\tilde{S}$ is

$$\begin{aligned}
&\frac{\partial \tilde{S}}{\partial z} + \frac{1-\nu^2}{z\nu}\frac{\partial \tilde{S}}{\partial \nu} + \frac{1-u^2}{zu}\frac{\partial \tilde{S}}{\partial u} - \frac{\nu^2+u^2}{z\nu^2u^2}\tilde{S} + \sigma\left(\frac{1}{\nu}+\frac{1}{u}\right)\tilde{S} \\
&\quad = \sigma\lambda\left[1 + e^{-2\sigma z\nu}H(\nu_c-\nu) + \frac{1}{2}\int_0^1 \tilde{S}(z,\nu,u')\frac{du'}{u'}\right] \\
&\qquad \cdot \left[1 + e^{-2\sigma zu}H(\nu_c-u) + \frac{1}{2}\int_0^1 \tilde{S}(z,\nu',u)\frac{d\nu'}{\nu'}\right], \qquad z > a, \\
&\qquad\qquad \tilde{S}(a,\nu,u) = 0.
\end{aligned} \tag{2.6}$$

This result, in different notation and variables, was first obtained by Bailey [4]. It may also be derived by direct substitution of (2.5) into (2.2), but the details of this derivation are lengthy and will not be given here.

3. Nonuniqueness of the System (2.2) and its Computational Implications

The system (2.2) for the total reflection function does not determine a unique solution. This is easily shown in the special case of no absorption, $\sigma = 0$; there are then two solutions

$$S^{(1)} = \delta(u-\nu)\,H(\nu_c-\nu) \tag{3.1}$$

$$S^{(2)} = 0. \tag{3.2}$$

Solution $S^{(1)}$ is the desired solution, describing the unattenuated passage of that radiation which does not intercept the core. Solution $S^{(2)}$ is a second, nonsingular solution.

In general when $\sigma \neq 0$, there will always be a nonsingular solution to (2.2) in addition to the desired singular one. This nonsingular solution may be determined, for example, by a power series expansion in the parameter $(z - a)/a$. This is essentially an expansion about the plane parallel case, for which singular solutions are absent, since any ray incident at a surface can never emerge again from that surface. The analytic nature of the expansion assures that the solutions to the spherical case so generated are also nonsingular. Also, the nonsingular solutions may be found by any numerical method involving polynomial or other smooth representations of S, which by their very nature exclude singular solutions. Both these methods were employed in [7], and it must be concluded that the numerical solutions found there are the nonsingular solutions of (2.2).

The crucial question to be considered is the identification of these nonsingular solutions. In the case $\sigma = 0$ the diffuse reflection function vanishes, and the nonsingular solution also vanishes. This suggests that the nonsingular solution may simply be the diffuse reflection function for all values of σ. Indeed this identification has apparently been made in [6, 7]. However, it may be shown that this is not the case. For if it were true it would imply that the same function $S \equiv \tilde{S}$ satisfies both Eqs. (2.2) and (2.6). This would mean that the right-hand sides of these equations are equal; but this cannot be true for all values of ν and u when $\sigma\lambda \neq 0$. For example, when $\nu = u < \nu_c$, there should be the identity

$$(X + e^{-2\sigma z\nu})^2 = X^2, \tag{3.3}$$

where

$$X = 1 + \frac{1}{2}\int_0^1 \tilde{S}(z, \nu, u')\,\frac{du'}{u'}. \tag{3.4}$$

But the exponential term is positive, as is X, since $\tilde{S}$ is nonnegative; therefore,

$$X + e^{-2\sigma z\nu} > X > 0 \tag{3.5}$$

and the square of this inequality violates (3.3).

It must be concluded that the numerical methods presented in [6, 7] for finding the diffuse reflection function are invalid.

4. Discussion

The numerical results presented in [6, 7] may actually be correct to graphical accuracy, because the cases treated there are not very severe. The maximum ratio of shell thickness to core radius considered was 3/50, and the optical thickness of the shell was 3. The additional exponential terms which distinguish (2.6) from (2.2) exceed 10^{-2} only for a very small range of ν, $(0 < \nu < 0.02)$, so that the results are probably accurate to within a few percent.

For future computations on this problem the system (2.6) would seem to be suitable, as the solution is an ordinary function, and presumably similar difficulties with uniqueness will not arise. Care should be taken in setting up representations of $\tilde{S}$ that some account be taken of the discontinuities which occur at values of ν and u equal to ν_c. However, even if one can avoid the difficulties of the system (2.2) by treating (2.6) instead, this is, to quote Bailey and Wing [4], "of only slight consolation if one has put considerable effort and perhaps large amounts of computer time into the study of an originally incorrect formulation."

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